Pendulum Accuracy, part 1 – Purity and Quality

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Introduction

I was re-reading some horology books [1] and thinking again about Q in pendulum clocks. It is clear to me that Q is at the core of pendulum accuracy – and many authors have said the same. Yet the topic of Q still creates controversy. So either there remains a flaw in our understanding of pendulum Q or perhaps there is a continuing problem with the way the topic is presented.

This article attempts a different approach to Q. We simply define another factor called "P" and show it is P (purity factor) and Q (quality factor) *together* that determine pendulum clock accuracy. This new explanation helped me understand what role Q plays, and does not play.

A couple of notes – this is intended as a non-technical, high-level discussion. I will use the word impulse instead of escapement. I may use the word accuracy at times when technically it should be called stability. And for simplicity let's assume the pendulum is impulsed every period.

Quick summary

Say **E** is the total energy of a pendulum and **ΔE** is the amount lost per period due to all forms of friction. ΔE is also the amount gained per period due to impulse. Realize that ΔE is probably not *exactly* the same, period to period to period, so let **σE** be the statistical variation in ΔE. Typically E is large, ΔE is small, and σE is smaller still. The claim here is that, ignoring unpredictable or uncorrected environmental effects, the stability of the pendulum is σE/E.

In addition to the familiar old *quality* factor **Q = E/ΔE**, we define a useful new *purity* factor **P = ΔE/σE**. While Q defines how much energy loss (and then, gain) occurs during normal pendulum operation, P defines how consistently that energy is managed. The claim here is that the stability of a pendulum is equal to 1/PQ.

This simple consideration of σE, ΔE, and E, as well as constants P and Q sheds light on the role of Q in a pendulum clock. In particular, it confirms that accuracy is directly related to Q. It also shows accuracy is directed related to P. It suggests the best accuracy is obtained when P times Q is the highest, not necessarily when Q alone is the highest or when P alone is the highest.

The purpose of this article is to look at the role energy stability plays in pendulum performance and to emphasize that a *pair* of factors, P (purity) and Q (quality), is responsible for pendulum accuracy, not just Q. This is not to undermine the importance of Q, but rather to clarify it.

The problem with Q

In horological books and journals, opinions on Q vary from "the higher the Q the better the accuracy" to "Q is likely a factor in pendulum accuracy" to "Q has little to do with pendulum accuracy". Matthys [1] begins his book with a long list of arguments for and against Q. In re-reading these now it's clear that there's a missing piece to the puzzle and the arguments come across more as questions begging for a simple explanation.

Many authors point to Bateman's classic chart of Q vs. accuracy as the most convincing evidence that Q affects accuracy. But the Q chart can be misinterpreted to imply total *causation* instead of partial *correlation*. This creates divisions within the horological community with regards to Q.

So my hope here with this treatment of energy is to unite the two camps with a simple, concise explanation. I will take the middle ground and both promote Q and yet explore the notion of P so that P and Q are equally appreciated partners in the quest for understanding pendulum accuracy.

Energy, delta

We call **E** the total energy of the pendulum. During every period some energy is lost due to friction. If designed correctly, every period that same energy is restored by an impulse. We call this amount of energy **ΔE** (delta E).

It's easy to calculate E since at all times E = KE + PE and we know KE = $\frac{1}{2}mv^2$ and PE = mgh. So a live measurement of v or h gives E. The typical way to measure v is with an optical gate.

It's also easy to measure ΔE. The typical way is to let the pendulum "run down" for a few tens or hundreds of cycles and watch E decay. ΔE is then calculated from an exponential fit. In general, ΔE is much smaller than E.

Energy, sigma

There is another energy measurement that is important. During every period there is ΔE loss of energy and ΔE gain of energy. If the pendulum is operating perfectly, the gain is the same, period after period. The loss is the same, period after period. And the loss equals the gain, period after period.

However, in a real pendulum, there are slight *variations* in the energy loss each period; friction (from all its sources) is not perfectly uniform. Similarly, there are slight variations in energy gain each period; impulse is not perfectly uniform – in time, in duration, or in magnitude. Finally, in a real pendulum the gain each period does not perfectly match the loss each period. Thus, every period there is some degree of energy mismatch. We will not distinguish among all the individual sources of random energy variation or fluctuation. Instead, statistically, we will simply call the total variation in energy **σE**. In general, σE is much smaller than ΔE.

If we made numerous measurements of the energy flow each period, ΔE is the *mean* and σE is the *standard deviation*. Intuition tells us that the larger E is, and the smaller ΔE and σE are, the better the pendulum will keep time. Variation in energy, however small, results in variation in rate, which when integrated results in variation and drift in time. So energy stability is desirable.

Ratio, Q

What can we do with our three energy values? The ratio E/ΔE is what we call Q, or quality-factor. Because it is a ratio of two similar units, Q is dimensionless. Notice that it is not *timeless*, in the sense that the numerical value for Q is scaled by the choice of time units. If periods are used Q = 2πE/ΔE, if swings are used Q = πE/ΔE, and if radians are used Q = E/ΔE. We could also talk about seconds, t, or sampling interval, τ, but let's not worry about scale factors now.

As an aside, we know the letter "Q" was chosen almost by accident and only later was the word *quality* or quality-factor associated with it. [2] [3] But it has a nice ring so the name stuck.

The notion of Q resonates well with our intuition. The quality factor appears in many different branches of science, from electrical engineering, to acoustics, to mechanics, to quantum physics. I think it's fair to say most people have a grasp of what Q is. And even if they don't – the formula for Q, as in $[2\pi]E/\Delta E$, is so simple you can use Q whether you understand it or not.

Ratio, P

There is one more ratio we can create. We know ΔE is the deliberate, tiny fraction of E that is consumed (and replenished) each period. Similarly, σE is the undesirable, random, tiny fraction of ΔE that occurs each period. Think of words like variation, instability, noise, or jitter.

If you were assigned to fill many bags of sand, then E is the sand pile, ΔE is the nominal size of each bag, and σE is how full each bag is: sometimes the bag isn't quite full, sometimes a bit too full. As a laborer you do not decide E or ΔE but your skill is reflected by your σE.

So let's define $P = \Delta E/\sigma E$. Like Q, we define P in a reciprocal manner so that bigger is better. It is also dimensionless. And while we're at it we should name it; something about consistency, or stability, or precision, or perfection – let's call it *purity*.

You have not heard of P before. I made it up as a way to clarify misunderstandings about Q. But as we shall see, the seeds of P have been present for a long time. It's just that without a cute name and symbol it's easy for people to overlook it, which then allows Q to steal all the focus.

The notion of P also follows our intuition. In many aspects of technology, consistent behavior is the goal. Even with clocks – we are less concerned about the right time, or even the right rate; what we want is *consistency* of rate. So a statistical measure of energy consistency is welcome.

Simple and symmetrical

To summarize, E is the total energy of the pendulum. Measure it very closely and you can calculate the mean, ΔE, and the standard deviation, σE, of energy gain/loss. Note σE is the combination of all possible sources: friction in support, suspension, and air drag, as well as impulse timing, duration, and power. So from these three energy values, it's easy to compute two ratios. $E/\Delta E$ is what we call Q and $\Delta E/\sigma E$ is what we call P. That's it.

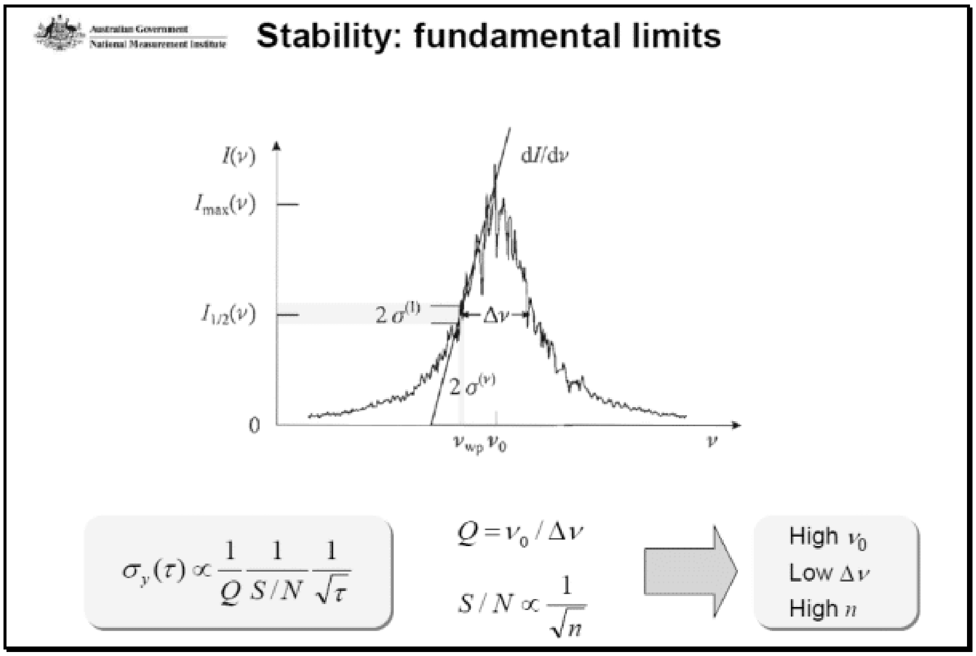
It seems obvious now that given E, ΔE, and σE then two ratios must exist. I like the idea of Q being a ratio that reflects the *small portion* of E that is involved each period. And I like the idea of P being a ratio that reflects the *small variation* in ΔE that inevitably occurs. We don't want to blame a constant like Q for small timing errors in a clock. That must be reflected in something else; that something else is P. For perspective, note that in pendulum clocks Q ranges from about 100 to 100,000 or even 1,000,000 and P ranges from about 10 to 1,000 or even 10,000.

The symmetry and simplicity of P and Q as derived from E, ΔE, and σE is striking. There must be something to this. But what do P and Q really mean? What does any of this have to do with pendulum accuracy? And why does everyone talk about Q and hardly anyone mentions P?

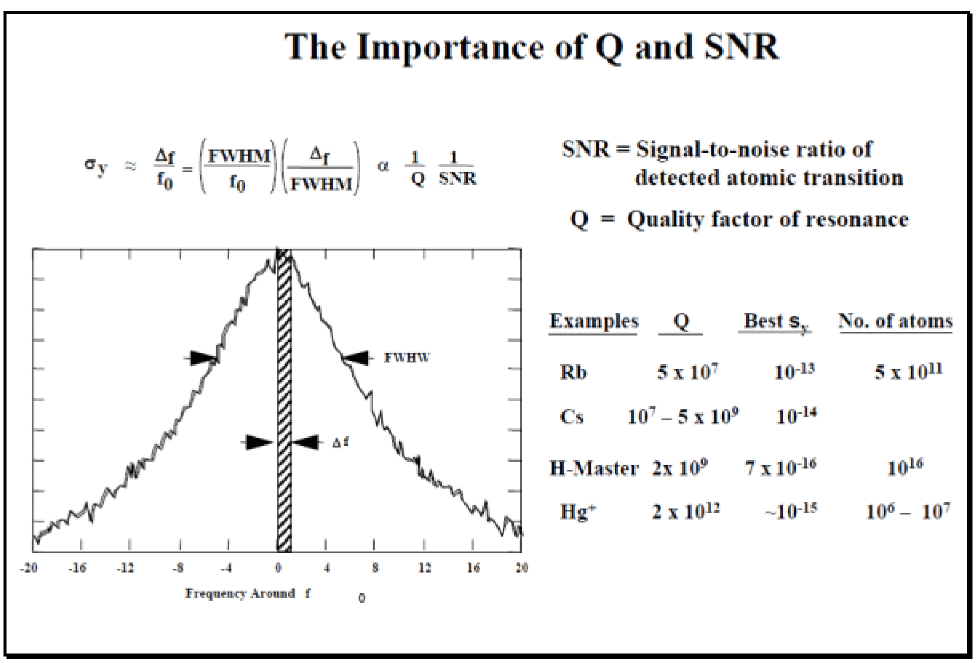
A hint from atomic clocks

When we read articles about Q we encounter nicely drawn, smooth bell curves and idealistic descriptions of resonance and stability. There is no mention of real-world noise and instability.

The mathematics and quantum mechanics of atomic clocks is very complex. But in spite of pages of equations the predicted performance of an atomic clock can be summarized with a very simple expression. The following are two examples from atomic clock tutorials on the web [4] [5]:



And also [6] [7]:



Immediately, several things strike me about these slides.

1. They do not draw the resonance as a nice smooth curve like you find in electrical engineering or horological texts about Q. Instead this is real-world stuff: the peak is not round; the edges are not smooth; the halves are not perfectly symmetrical; the sides do not reach zero, and there is low-level random noise everywhere.
2. They speak of FWHM (full-width, half-maximum) and Δf, clearly showing that Δf can be smaller, and maybe, *much smaller* than FWHM.
3. They clearly show that stability is directly proportional to 1/Q. That is, the higher the Q the better the stability.
4. They also clearly show that stability is proportional to something else – the signal-to-noise ratio (SNR, or S/N).
5. They use standard deviation or Allan deviation, but use proportional-to instead of equals.
6. The stability is proportional to the square root of tau, the sampling interval. This means the model predicts WFM (White Frequency Modulation) noise, which goes down by slope -½ on an ADEV (Allan Deviation) plot.

Similar examples can be found in books about modern atomic timekeeping.

Accuracy (stability) is 1/PQ

And so, it occurred to me that an equation for pendulum stability could be as simple as:

Pendulum stability = 1 / (P×Q)

Here Q is our old friend (Q = E/ΔE) and P is our new friend (P = ΔE/σE), which essentially is a S/N ratio. This means pendulum accuracy and stability is a function of its *purity* and *quality*.

The more you think about it the more sense this makes. Q reflects the static, intended design of the pendulum (mass) and initial operating conditions (amplitude) and nominal energy loss, while P reflects the dynamic, undesirable behavior; the variations in energy each period. Q determines what fraction of energy goes in and out of every interval and P reflects how well that fraction of energy is managed. It is intuitive that both factors play a role in the accuracy of a pendulum clock. You can't have one without the other.

Given this new understanding, that the accuracy of a pendulum clock is 1/PQ, how do we make a clock better? The obvious answer is to increase Q, or increase P, or increase both.

The less obvious answer is that there may be cases where one can still gain performance by increasing one factor a lot even if the other factor decreases a little. If it is possible to make a design change that increases Q by 50% and P decreases by 20% as a result, the net gain in PQ is 25%. Similarly, if one made a design change that increased P by 10× at the cost of 5× lower Q, the new gain in PQ is still 2×. In other words, to improve performance one should always concentrate on PQ, not just P or not just Q.

Note that the Q-factor of the textbook electrical engineering world doesn't need a P-factor – which explains why simple analogies carried from engineering fail to convince horologists.

Interesting observation

We know that increasing Q or increasing P is good for pendulum performance since:

Accuracy = 1/PQ

Now let's substitute for **P** and **Q**,

Accuracy = 1/(PQ) = 1 / (**ΔE/σE**)(**E/ΔE**) = 1 / (E/σE/) = σE/E

This is both expected and unexpected. The fact that "accuracy" or fractional frequency stability, which is often denoted Δf/f or σ(τ) or ADEV(τ) would be equal to the fractional energy stability makes sense. In physics, energy and frequency go hand in hand.

What is at first surprising is that <u>neither P nor Q</u> appears in this equation! But it makes sense when we remember that both P and Q are defined in terms of ΔE, one with ΔE as numerator and one with ΔE as denominator. The conclusion is – that at its core, the stability of a pendulum is not due to P or Q at all, but simply the amount of energy fluctuation relative to the total energy.

So accuracy is a simply a function of σE and E. But because ΔE must exist in any real pendulum clock and have some numerical value, then P and Q will exist and have some value. Perhaps P and Q can be better thought of as *derived* values, while energy is more fundamental.

The good news is this frees the pendulum clock designer from having to optimize P or optimize Q but instead do whatever it takes to minimize σE. Clearly there are a myriad of design choices made during the development and fabrication of a pendulum clock. If every choice is made so that σE is as small as possible and E is as large as possible, then the best accuracy will be obtained. By contrast, decisions to optimize P alone or Q alone are probably not ideal.

What happens then is that P and Q are calculated, numerical side-effects of real, practical energy decisions. Thus it seems to me the recipe for an accurate pendulum clock is:

1. Choose a value of ΔE to make σE as small as possible (large P)
2. Make E as large as possible (large Q).
3. Do not optimize for P. Do not optimize for Q. Optimize only for P×Q.

In other words, increasing Q is not the goal. A large Q may be a side-effect, but it should not be the goal. If the clock ends up being more accurate with smaller σE and larger P but smaller Q, there is no need to worry. The goal is the highest accuracy, not the highest Q.

I say this because it is not inconceivable that a well-designed, well-made pendulum clock might have unusually high P in spite of low Q. They key to performance is the product P×Q. Note that mathematically at least, ΔE is free to move up or down and P×Q will not change.

More to come (in part 2)

In absence of direct physical measurement of σE, one can simply compute P knowing stability and Q. In the next article, we will look at numerical values of P and Q and address the issue of constants such as, (to paraphrase Shakespeare) 2π or not 2π.

We will look at the Allan deviation (ADEV) and the role that tau (τ) plays.

Finally, we'll examine the classic 1977 Bateman graphs of Q that conclusively demonstrated that Q was a linear factor in pendulum accuracy and see how the idea of P and Q are consistent with his "phase stability" mathematics.

Conclusion

I hope what is presented here clarifies your understanding of pendulum performance and Q. I realize it's odd to add a new variable to help explain an old variable but it seemed Q was overloaded with too much meaning and made to carry more responsibility that it should have. By stripping Q back to its simple definition of an energy fraction, and by defining P to reflect energy fluctuation, we end up with two simple constants. [8]

This notion of P and Q helps put to rest every "Q argument" that I've heard. And if there are indeed two camps regarding the role of Q, the 1/PQ expression exonerates both camps. Yes, Q is important. No, Q is not the only thing. P times Q is the key. More Q is better. More P is also better. More PQ is the best, even if P alone or Q alone is lower. Higher E is better; lower σE is better; there is wide latitude in ΔE since it doesn't directly affect performance.

Even though Q is likely much larger than P in many cases [9], I believe there would be less confusion if every mention of pendulum Q (*quality*-factor) also mentioned P (*purity*-factor). They are joint partners in the quest for better understanding and better pendulum clock accuracy.

References

[1] A. J. Rawlings, "The Science of Clocks and Watches", 1993, 3rd edition, the one with all the commentary by Philip Woodward and Douglas Bateman, et al. Highly recommended.

Robert Matthys, "Accurate Pendulum Clocks", 2004

Gregory Baker, James Blackburn, "The Pendulum: a case study in Physics", 2006.

Philip Woodward, "Woodward on Time", 2006

[2] K.S. Johnson is credited with naming Q and choosing reciprocal notation. See "The Story of Q" by E. I. Green, a copy of which is found online:

http://www.collinsaudio.com/Prosound_Workshop/The_story_of_Q.pdf

[3] Here's an excerpt from US patent 1,628,983 where I'm told Q is first mentioned:

The transmission losses in such a structure will depend (1) upon the ratio Q of the reactance to the resistance of the coils and (2) upon the frequency—although all frequencies from zero (or D. C.) up to infinity will pass through the structure. The losses in the structure will also depend upon the ratio of the impedances $\left(\frac{R_1}{R_2}\right)$ between which the structure is to work.

The following approximation formula has been found useful in determining the effect of the ratio Q of the reactance of the coils to their effective resistance as well as the effect of the ratio of the impedances between which the structure is going to work:

$$\text{“transformer loss”} := \frac{\left(\sqrt{\frac{R_1}{R_2}}+\sqrt{\frac{R_2}{R_1}}\right)(1+K)}{.4Q} \qquad (24)$$

[4] PPT slide by Bruce Warrington, "Next-generation frequency standards", page 9.

http://www.ieee-uffc.org/frequency-control/learning/pdf/Warrington-Next-generation_frequency_standards.pdf

[5] Fritz Riehle, "Frequency Standards Basics and Applications", page 77.

[6] PPT slide by Lute Maleki, "Progress in Atomic Clocks and Tests of Fundamental Laws", page 5.

http://www.ligo.caltech.edu/~veronica/CaJAGWR/info/general/maleki.pdf

[7] See also the PPT slide by Robert Lutwak, "Principles of Atomic Clocks", page 7.

http://www.ieee-uffc.org/frequency-control/learning/pdf/Lutwak_AtomicClocks_Tutorial_pdf.pdf

[8] I went through Rawlings page by page and tallied the use of variables from A to Z. Especially because pendulum period is usually denoted by T, the letter P was one of the least used. So I picked that letter to denote the missing pendulum factor. Once P was chosen I thought of words it could stand for. Performance, perturbations, precise, and perfection came to mind, but *purity* hit the spot. My thought was most pendulum clock enthusiasts can handle a concept like purity but SNR would require mathematics or engineering tangents about signal and noise theory and that would distract from the simple concept that P is trying to convey.

[9] The impact of SNR and Q on an atomic clock is profound. The resonance width of a typical commercial cesium clock is about 500 Hz. The Q is then 9192 MHz / 500 Hz = 20 million! The SNR (at 1 Hz) is about 5000. So the stability (τ 1 s) is about 1/5000•20000000 or 1e-11. Note that log10 (5,000) is 3.7 and log10 (20,000,000) is 7.3. So roughly ⅓ of the stability is due to high SNR and ⅔ is due to high Q.

Since this is WFM (white frequency modulation) the stability improves by √τ, so by 2 weeks (1 million seconds) stability is down to the 1e-14 level. In this case, roughly ¼ of this is due to high SNR, ½ is due to high Q, and ¼ is due to long integration time. Yes, you can measure atomic clock Q at home: http://leapsecond.com/pages/cspeak/

[10] Copies of my HSN pendulum papers: http://www.leapsecond.com/hsn2006/

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