"Q"

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Ken Smith looks at the origins and subsequent applications of this often misunderstood ratio.

While talking to a student using the Q-meter the other day, I was reminded of the vast field of applications in which this un-named letter now appears. I asked the student what he was measuring He replied that Q was something to do with how 'good' the coil was – but looked blankly at me when I said, "What about its application to the width of spectral lines?"

I was hardly being fair, of course. Q has to do with the characteristics of spectral lines *and* inductors, *and* tuned circuits... But Q also appears when piano strings, quartz resonators, motor cars, auditoriums, bouncing balls and, among other things, the earth itself are discussed. The student was studying electronics engineering, which these days hardly results in any breadth of education at all, so how was he to know?

The odd thing about Q is that it is just... Q. There is no named quantity which is then given the symbol. It turns out that Q measures a number of things, all of which are often only partly understood. Here again we have an example of a fuzzy area often treated badly in lecture courses and so on.

This topic has developed into a little flurry of interest again recently in *Electronics and Wireless World*, via letters from august institutions in Cambridge enquiring about it. Lady Jeffreys[1,2] and D. McMullan[3,4] made the enquiries and also wrote short papers in the Royal Astronomical Society's *Quarterly Journal*. Geophysicists appear to have discovered Q through the work of the microwave engineers during and after the second world war. Further, it seems these colleagues of ours like to use the reciprocal of Q, but as I show a little later, this is the dissipation factor, and is already available as such. I was surprised to see a reference in the cited literature that someone had called Q the dissipation factor – an error, of course.

It is undoubtedly true that Nikola Tesla was aware of circuit magnification when he exploited his enormous LC ratios to produce kilo and megavolt r.f. intensity levels on his Tesla Coil secondaries.

Origins

The question of origin of this un-named symbol is a little obscure. I have adopted the evidence that Q first appeared in the notebooks of K.S. Johnson, who was working in the Western Electric Co. engineering department, USA, at the time of the first world war. Johnson appeared to be aware of the importance of the ratio, 'coil reactance to resistance' as early as 1914, but labelled it K. By 1920 he was using Q for this ratio and said when asked why Q? "Well, it does not stand for 'quality factor' but since all the other letters are so overworked, I only had Q left." He certainly published many references to Q in his book[5] on telephone engineering published in 1924/25.

At this stage, Q was a dimensionless factor with no precise name, but it was defined as

$$Q=\frac{\omega L}{R}$$

Engineers well before this time had utilized the rate of decay of a wave train, or the *logarithmic decrement* δ, on the wireless communications side and the *power factor* cosφ, in heavy engineering circles. *Dissipation factor* D, was also widely used and it was apparently the 'upside down' or psychologically dissatisfying nature of D that prompted Johnson to use its reciprocal and ultimately to label it Q. D becomes larger as the losses, or performance, gets worse. The dissipation factor also tends to be a small fractional number in light current work. Its reciprocal Q, is an integer or large integer and grows as the quality of performance grows. Although Johnson denied 'quality factor' was ever in his mind, nevertheless V.E. Legg popularized this possible meaning and it has stuck.

Very quickly Q was seen to relate not only to D but to δ and cosφ also. As an interesting sideline, the log.dec. δ was very important in the days of spark transmission, with its damped wave trains. But as c.w. narrow-band techniques took over, it faded into insignificance, (back to the laboratory where a few ballistic galvanometers were still employed).

What is Q

Figure 1 shows an ordinary phasor diagram that, in this example, applies to a lossy inductor. From the phasor geometry, the power factor is

$$\cos\phi=\frac{R}{Z}$$

The dissipation factor is

$$D=\frac{R}{\omega L}=\tan\psi$$

Q is simply the reciprocal of D

$$\therefore Q=\frac{\omega L}{R}=\cot\psi$$

Of some interest is to note that D is proportional to the *dissipated power* in the circuit (i^2R), while Q is proportional to the *stored energy*, ($\frac{1}{2}Li^2$).

Another interesting point is that power engineers strive to maximize the dissipation, in other words, they try to arrive at a zero phase angle, giving

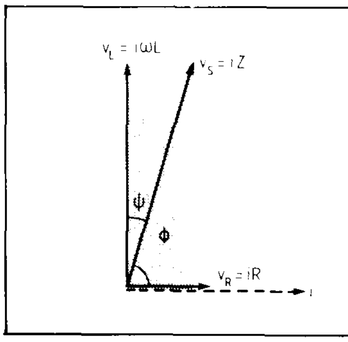

Fig. 1. The current phasor i is common to L and R in the series arrangement shown. V_R is in phase with i, while v_L leads i by 90°. The phase angle between i and v_s is used for power factor calculations. ($\psi = 90° - \phi$) is often called the loss angle.

$\cos\phi = 1$; while communications engineers want to minimize losses, i.e. make ψ as small as possible, which is the same thing as obtaining large Q.

Q factors for inductors tend to be much lower than those attained in good-quality capacitors. Therefore, the coil losses in a resonant circuit predominate. A major step forward was realised when it was found that Q could be applied to a tuned circuit at its particular resonant frequency f_0, as well as meaning a figure of merit for an inductor at any frequency. A number of results followed from this. The tuned circuit in Fig. 2 stores energy in its reactances and dissipates it as heat in the resistance.

"Well, it doesn't stand for 'quality factor', but since the other letters are so overworked, I only had Q left." – K.S. Johnson.

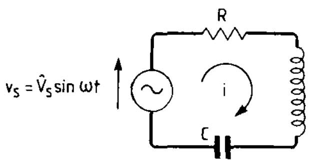

Fig. 2. As v_s drives current through the circuit, the stored energy alternates between the capacitor containing it all and the inductor storing it as magnetic energy. R dissipates some energy as heat during every cycle.

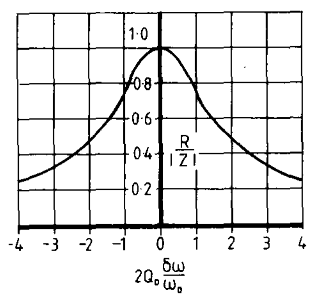

Fig. 3. The response curve of a tuned circuit is the well known bell-shaped curve. In the normalized form shown here, it applies to any series LCR circuit and the larger Q becomes, the narrower the peak between the points which are 0.707 times down.

The stored energy oscillates between the magnetic field of the inductor and the electric field in the capacitor.

$\therefore$ energy stored $= \frac{1}{2}L\hat{I}_s^2$ joules

(Alternatively, energy stored could be written, $\frac{1}{2}C\hat{V}_s^2$.) The average power lost in the resistor is

$$\frac{\hat{I}_s^2 R}{2},$$

because the r.m.s. current

$$I_s = \frac{\hat{I}_s}{\sqrt{2}}.$$

Therefore the energy lost per cycle is

$$\frac{\hat{I}_s^2 R}{2f}$$

where f is the frequency in hertz.

We are now in a position to examine the relationship of the energy stored to that dissipated per cycle. The ratio is

$$\frac{\frac{1}{2}fL\hat{I}_s^2}{\frac{1}{2}RI_s^2} = \frac{fL}{R} = \frac{2\pi fL}{2\pi R} = \frac{\omega L}{2\pi R} = \frac{Q}{2\pi} \qquad (1a)$$

This is now a more fundamental definition of Q than Johnson's original, as it is based on energy relationships.

$$Q = 2\pi \frac{\text{total energy stored in periodic system}}{\text{energy dissipated in one period}} \qquad (1b)$$

Since in the tuned circuit at resonance

$$\omega = \omega_0 \text{ and } \omega_0 L = \frac{1}{\omega_0 C}'$$

the current is

$$I_s = \frac{V_s}{R}$$

and is a maximum. This current flows through the inductor (and the capacitor...) and the voltage drop across L is

$$V_L = I_s\omega L = \frac{V_s\omega L}{R} = QV_s$$

This shows that the voltage across L (or across C) is Q times the voltage generator value. With Q of some hundreds, V_L can be large. This explains why Tesla managed such large 'magnification factors'.

Very similar relationships can be derived for a parallel tuned circuit, which is the dual of the series one I have treated.

Detuning: selectivity and bandwidth

A look at the series tuned circuit again shows that, off-tune, it appears to offer an impedence given by

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

On tune, $\omega = \omega_0$ and

$$\omega_0 L = \frac{1}{\omega_0 C},$$

$\therefore \frac{1}{C} = \omega_0^2 L$, so that Z can be written

$$Z = R + j\omega_0 L\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$= R\left[1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

Q_0 is the *unloaded* Q of the circuit at frequency ω_0.

If the Q is large, then the variations of ω about ω_0 are fractionally very small to maintain significant results. (In other words, resonance is over and done with very quickly if Q is large.) This means can be written $\omega_0 \pm \delta\omega$ where $\delta\omega$ is very small. Inserting $\omega_0 + \delta\omega$ into

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}$$

gives

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \doteq \frac{2\delta\omega}{\omega_0}$$

by neglecting $(\delta\omega)^2$ in comparison to other terms.

$$\therefore Z = R\left[1 + j2Q_0\frac{\delta\omega}{\omega_0}\right]$$

By considering the absolute value, or modulus of Z, some further interesting results follow.

$$\therefore |Z| = R\sqrt{1 + 4Q_0^2\left(\frac{\delta\omega}{\omega_0}\right)^2}$$

A plot of $R/|Z|$ against $2Q_0\delta f/f_0$ (the 2π having cancelled) gives a normalized universal resonance curve applicable to all tuned circuits, and by implication, to all resonant systems, see Fig. 3.

When $4Q_0^2(\delta f/f_0)^2 = 1$ the magnitude of the reactance of the circuit equals the resistance, and the amplitude of the current drops to $1/\sqrt{2}$ of the value at f_0, which means that the power dissipated drops to one half the value at resonance. This yields another significant relationship for Q,

$$Q_0 = \frac{f_0}{2\delta f}$$

$2\delta f$ can be written Δf, the "half power bandwidth", and the ratio of f_0 to $2\delta f$ is a measure of *sharpness of reasonance* or selectivity.

Damping, the logarithmic decrement

A tuned circuit, or a church bell, a tuning fork, a motor car on its springs and (if they are not too good!) shock absorbers – all oscillate with an exponential decay, or ringing of the type shown in Fig. 4.

A solution of the equations for these natural oscillations is, (for the LCR circuit)[6],

$$i(t) = \hat{I}_s e^{-\left(\frac{R}{2L}\right)t} \cos\omega_0' t$$

where ω_0' is nearly equal to the ω_0 used previously.

From Figure 4 you can see that the time to go from one peak value to the next is $t_2 - t_1 = T$ and if the two peak currents corresponding to t_1 and t_2 are I_1 and I_2, then

$$\frac{\hat{I}_2}{\hat{I}_1} = e^{-\frac{R}{2L}(t_2 - t_1)} = e^{-\frac{RL}{2L}} = e^{-\delta}$$

where δ is known as the *logarithmic decrement*[7]

$$\therefore \delta = \frac{RT}{2L},$$

$$\text{or because } T = \frac{1}{f},\ \delta = \frac{R}{2fL}$$

$$\therefore \delta = \frac{\pi}{Q}, \text{ or } Q = \frac{\pi}{\delta}$$

This shows that the oscillating current in a circuit with a Q of 100 dies away to 37% of its initial value after about 32 cycles. (It also shows that a car suspension Q of 100 would cause the car to oscillate the same number of times after a bump in the road, and so high Q is not always sought!)

Loaded, unloaded and external Q

In Figure 5, the tuned circuit is driven from a voltage generator whose internal resistance is R_s. Now, from the definition of Q as 2π times the ratio of energy stored to energy leakage, as given in equation (1), the sinks of energy loss have been simple up until now. There was only one, the coil resistance. But in Fig. 5, R_s will also dissipate energy.

$$\therefore Q_L = \frac{\omega_0 L \hat{I}_s^2}{2} \Big/ \frac{\hat{I}_s^2 (R=R_s)}{2}$$

$$= \frac{\omega_0 L}{R+R_s} \qquad (2)$$

Q_L is the *loaded Q*, and is the Q of the entire system. The reciprocal of equation (2) gives,

$$\frac{1}{Q_L} = \frac{R}{\omega_0 L} + \frac{R_s}{\omega_0 L}$$

$$\therefore \frac{1}{Q_L} = \frac{1}{Q_o} + \frac{1}{\omega_{Qex}}$$

In this equation, Q_o is the *unloaded* Q of the tuned circuit only: Q_{ex} is the *external* or *radiation* Q.

In a transmitter, for example, Q_o should be large and therefore $1/Q_o$ very small, so that although the loaded Q may be low, nearly all of it is radiation Q, (and the r.f. energy is not warming up the inductor very much).

Q is everywhere

The use of Q rapidly proliferated. *Any* periodic system was said to have a Q if it stored energy and dissipated it over time. From equation (1) a tennis ball can be tested for its Q, and there would be a relationship with its coefficient of restitution. Are we one day going to see advertisements from the makers extolling their "High-Q tournament balls..."? If wild claims were made, such as balls with Q of 1000, they would have to explain how their ball should still be rising to 37% of its initial height after bouncing 320 times. Actual balls possess Q of ~ 6 or 7. Table 1 gives a diagrammatic list of some typical values.

Even the earth, as a periodic system, has a Q. It was this very high Q that timed the world's clocks until recently – when higher precision atomic clocks took over, (the caesium clock). The earth has an irregular rotation period, mainly due to the seasonal sap rising and falling in the vegetation, but it has a monotonic decline in its period of around 1.64×10^{-3} per century. The energy stored in the earth's rotation is $\frac{1}{2}I\omega^2$ joules: I is its moment of inertia, ω the angular velocity. Therefore, from the energy lost in one period compared to 2 times the energy stored, the earth's Q is $\sim 10^{13}$.

The ubiquitous Q is a remarkable idea that, although represented by just a single letter from the alphabet, has grown from Johnson's original ratio of reactance of a coil to its resistance, to encompass the vibrations of atoms on the one hand, to the rotation of the earth itself on the other. One day perhaps we shall hear about the Q of the Galaxy. . . .

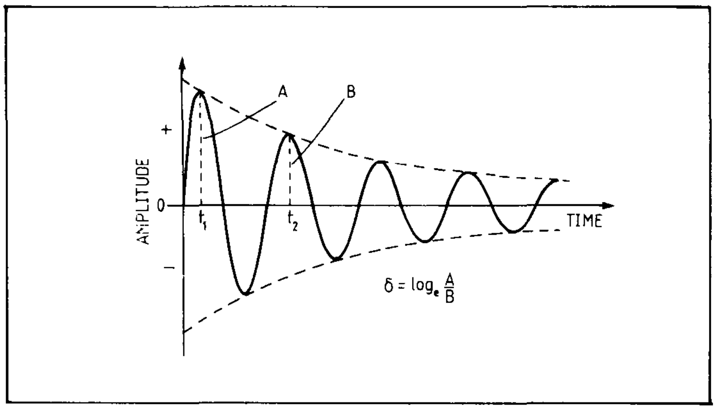

Fig. 4. If the drive v_s is suddenly turned off in Fig. 2, then the energy is gradually dissipated and the damped oscillation, or "ringing" shown is the result. The logarithmic decrement is defined from the rate of decay.

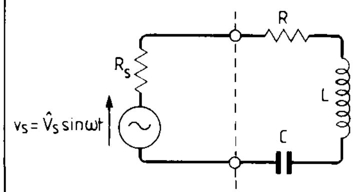

Fig. 5. Some energy per cycle is dissipated internally in R. Further energy appears as heat in R_s. The loaded and unloaded Q definitions arise from this division of labour.

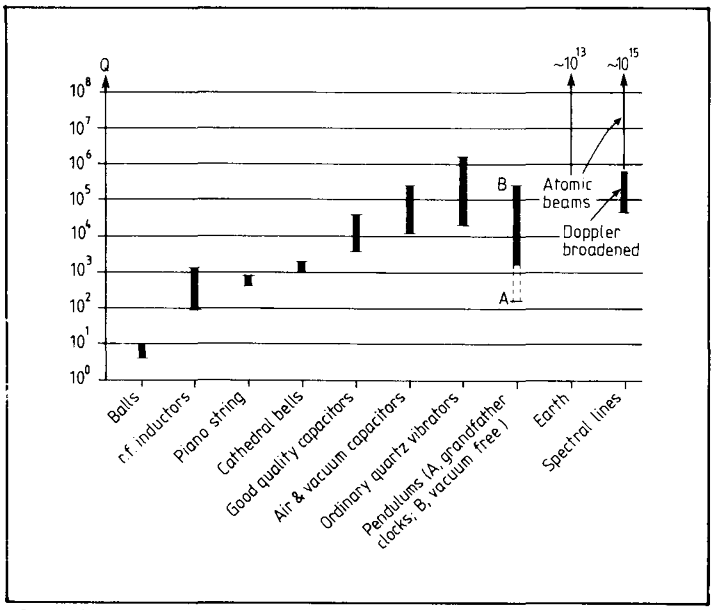

Fig. 6. Shown here is a schematic representation of the Q of some periodic systems and devices met in technology and everyday life.

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